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Erdős–Turán Type Inequality for the Sum of Successive Fundamental Polynomials of Hermite Interpolation[1]

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Let ℓ_k, $1 \leqslant k \leqslant n$, be the fundamental polynomials of Lagrange interpolation on the nodes $x_n < x_{n-1} < \cdots < x_1$. The classical Erdős–Turán inequality is

$$\ell_k(x) + \ell_{k+1}(x) \geqslant 1, \qquad x \in [x_{k+1}, x_k], \quad 1 \leqslant k \leqslant n-1.$$

This paper gives an extension for such an inequality to the sum of successive fundamental polynomials of Hermite interpolation.

1. INTRODUCTION AND MAIN RESULT

This paper deals with the Erdős–Turán type inequality for the sum of successive fundamental polynomials of Hermite interpolation.

Let $\mathbf{N}_1 = \{1, 3, 5, ...\}$, $\mathbf{N}_2 = \{2, 4, 6, ...\}$, $\mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_2$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and

$$-\infty \leqslant a = x_{n+1} < x_n < \cdots < x_1 < x_0 = b \leqslant +\infty. \tag{1.1}$$

We introduce the notation:

$$\langle a, b \rangle = \begin{cases} [a, b], & a > -\infty, \quad b < +\infty, \\ (a, b], & a = -\infty, \quad b < +\infty, \\ [a, b), & a > -\infty, \quad b = +\infty, \\ (a, b), & a = -\infty, \quad b = +\infty. \end{cases}$$

Let μ be a nondecreasing function on $\langle a, b \rangle$ with infinitely many points of increase such that all moments of $d\mu$ are finite. We call $d\mu$ a measure. Let $m_0, m_{n+1} \in \mathbf{N}_0$, $m_k \in \mathbf{N}$, $k = 1, 2, ..., n$, and $N := \sum_{k=0}^{n+1} m_k - 1$. We always

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assume that $m_0=0$ if $a=-\infty$ and $m_{n+1}=0$ if $b=+\infty$. Denote by $\mathbf{P}_N$ the set of polynomials of degree at most N and by A_{jk} the fundamental polynomials for Hermite interpolation, i.e., $A_{jk}\in\mathbf{P}_N$ satisfy

$$A_{jk}^{(i)}(x_q)=\delta_{ji}\delta_{kq}, \qquad i=0,1,...,m_q-1, \quad j=0,1,...,m_k-1,$$
$$q,k=0,1,...,n+1. \tag{1.2}$$

In particular, for Lagrange interpolation ($m_0=m_{n+1}=0$, $m_k=1$, $k=1, 2, ..., n$) we accept the notations $\ell_k := A_{0k}$, $k=1,2,...,n$, and in this case we have the classical Erdős–Turán inequality [2, Lemma IV]

$$\ell_k(x)+\ell_{k+1}(x)\geqslant 1, \qquad x\in[x_{k+1},x_k], \quad 1\leqslant k\leqslant n-1. \tag{1.3}$$

This inequality has many applications, say, it is used to estimate lower bounds for the Lebesgue function of Lagrange interpolation [5]. A weighted form of (1.3) is obtained by D. S. Lubinsky [4]. The main aim of this paper is to give a very general extension of this inequality to Hermite interpolation. To this end let

$$s_{-1}=1, \qquad s_r=(-1)^{m_0+m_1+\cdots+m_r}, \qquad r=0,1,...,n+1, \tag{1.4}$$

and

$$\sigma(x)=\operatorname{sgn}\prod_{k=0}^{n}(x-x_k)^{m_k}. \tag{1.5}$$

Then we can state the first main result as follows.

THEOREM 1. *Let $0\leqslant r\leqslant n+1$ and let an $(N+1)$th continuously differentiable function f satisfy*

$$f^{(j)}(x)\geqslant 0, \qquad x\in\langle a,x_r\rangle, \quad j=0,1,...,N+1. \tag{1.6}$$

Then

$$s_r\sigma\left[f(x)-\sum_{k=r+1}^{n+1}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\right]\geqslant 0, \qquad x\in\langle a,x_r\rangle, \tag{1.7}$$

$$s_r\sigma(x)\sum_{k=r+1}^{n+1}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\geqslant 0, \qquad x\in\langle x_r,b\rangle, \tag{1.8}$$

$$s_{r-1}\sigma(x)\left[f(x)-\sum_{k=r}^{n+1}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\right]\geqslant 0, \qquad x\in\langle a,x_r\rangle, \tag{1.9}$$

and

$$s_{r-1}\sigma(x)\sum_{k=r}^{n+1}\sum_{j=0}^{m_k-1} f^{(j)}(x_k)\,A_{jk}(x)\geqslant 0, \qquad x\in\langle x_r, b\rangle. \tag{1.10}$$

This is a very general and useful result whose consequences and applications are stated in the last section. The proof of this theorem is given in the next section. We need a fundamental lemma [3, Lemma I.5.3, p. 30], in which $Z(f, I)$ denotes the number of zeros of the function f in the interval I counting multiplicities.

LEMMA A. *Given a* $\xi\in\langle a, b\rangle$ *let an* $(N+1)$*th continuously function* f *satisfy*

$$f^{(j)}(x)>0, \qquad x\in\langle a, \xi\rangle, \quad j=0, 1, ..., N+1,$$

and let $P\in\mathbf{P}_N$, $P\neq 0$. *Then*

$$Z(f-P, \langle a, \xi\rangle)+Z(P, \langle \xi, b\rangle)\leqslant N+1.$$

2. PROOF OF THEOREM 1

It suffices to show (1.7)–(1.10) for the case when $a>-\infty$ and $b<+\infty$. In fact, if, say, $a=-\infty$ and $b<+\infty$, we can choose an arbitrary point $-\infty<A<x_n$. The inequality (1.6) implies that

$$f^{(j)}(x)\geqslant 0, \qquad x\in[A, x_r], \quad j=0, 1, ..., N+1.$$

Then that the inequality

$$s_r\sigma(x)\left[f(x)-\sum_{k=r+1}^{n+1}\sum_{j=0}^{m_k-1} f^{(j)}(x_k)\,A_{jk}(x)\right]\geqslant 0, \qquad x\in[A, x_r],$$

holds for every A implies the inequality (1.7). So does for the inequality (1.9).

Now let $a>-\infty$ and $b<+\infty$. By the definitions (1.4) and (1.5) we observe that

$$\sigma(x)=s_r, \qquad x\in(x_{r+1}, x_r), \quad 0\leqslant r\leqslant n. \tag{2.1}$$

First, we are going to show (1.7)–(1.10) for f satisfying

$$f^{(j)}(x)>0, \qquad x\in[a, x_r], \quad j=0, 1, ..., N+1, \tag{2.2}$$

instead of (1.6) (because we intend to use Lemma A). We separate two parts.

Part 1. The proof of (1.7) and (1.8). In this case put

$$P(x)=\sum_{k=r+1}^{n+1}\ \sum_{j=0}^{m_k-1} f^{(j)}(x_k)\, A_{jk}(x) \tag{2.3}$$

and distinguish three cases.

Case 1.1. $r=n+1$. In this case (1.7) and (1.8) are trivial, because $P=0$.

Case 1.2. $r=n$ and $m_{n+1}=0$. In this case (1.8) is trivial and (1.7) becomes

$$s_n\sigma(x)\, f(x)\geqslant 0, \qquad x\in[a, x_n],$$

which by (2.1) and (2.2) is true.

Case 1.3. $r<n$ or $m_{n+1}>0$. In this case we break the proof into a series of claims.

Claim 1. For the function

$$F(x)=\begin{cases} f(x)-P(x), & x<x_r,\\ P(x), & x\geqslant x_r,\end{cases} \tag{2.4}$$

we have

$$Z(F, (x_{r+1}, x_r))=Z(P, (x_{r+1}, x_r))=0. \tag{2.5}$$

In fact, the definition of F shows that for each k, $0\leqslant k\leqslant n+1$, the point x_k is a zero of m_k multiplicities of F. If the first equality of (2.5) was not true, by Lemma A we would have

$$\begin{aligned} N+1 &\geqslant Z(f-P, [a, x_r])+Z(P, [x_r, b])\\ &= Z(F, [a, x_{r+1}])+Z(F, (x_{r+1}, x_r))+Z(F, [x_r, b])\\ &\geqslant N+2,\end{aligned}$$

a contradiction. This proves the first equality of (2.5). Similarly, if the second equality of (2.5) is not true, by Lemma A it would lead to a contradiction:

$$\begin{aligned} N+1 &\geqslant Z(f-P, [a, x_{r+1}])+Z(P, [x_{r+1}, b])\\ &= Z(F, [a, x_{r+1}])+Z(P, (x_{r+1}, x_r))+Z(F, [x_r, b])\\ &\geqslant N+2.\end{aligned}$$

Claim 2. If $r\geqslant 1$ or $m_0>0$, then

$$F(x)>0, \qquad P(x)>0, \qquad x\in(x_{r+1}, x_r). \tag{2.6}$$

In fact, since $P(x_{r+1}) = f(x_{r+1}) > 0$ and $P(x_r) = 0$, (2.6) follows from (2.2) and (2.5).

Claim 3. The function σF does not change sign in $[a, b]$.

In fact, suppose to the contrary that the function σF changes sign at $z \in (a, b)$. If $z \notin \{x_1, ..., x_n\}$ then $F(z) = 0$. If $z = x_k$, $1 \leqslant k \leqslant n$, then the inequality

$$[\sigma(x_k - \delta) F(x_k - \delta)][\sigma(x_k + \delta) F(x_k + \delta)] < 0$$

would occur for all shall $\delta > 0$, which by (1.5) implies that the inequality

$$(-1)^{m_k} F(x_k - \delta) F(x_k + \delta) < 0 \tag{2.7}$$

would occur for all small $\delta > 0$. If we agree that

$$F^{(j)}(x) = \begin{cases} f^{(j)}(x) - P^{(j)}(x), & x < x_r, \\ P^{(j)}(x), & x \geqslant r, \end{cases}$$

then we can conclude

$$F^{(m_k)}(x_k) = 0. \tag{2.8}$$

Indeed, if $k \neq r$ then (2.8) is trivial; if $k = r$ then by (2.6) the inequality (2.7) means

$$(-1)^{m_r} P(x_r - \delta) P(x_r + \delta) < 0$$

and hence (2.8) with $k = r$ follows. This proves (2.8). By (2.8) we have $Z(F, [a, b]) \geqslant N + 2$, contradicting Lemma A. This contradiction proves Claim 3.

Claim 4. We have

$$s_r \sigma(x) F(x) \geqslant 0, \qquad x \in [a, b]. \tag{2.9}$$

We separate two cases.

Case 1.3.1. $r \geqslant 1$ or $m_0 > 0$. In this case by (2.6) and (2.1) $\operatorname{sgn}[\sigma(x) F(x)] = \sigma(x) = s_r$ holds for $x \in (x_{r+1}, x_r)$, which by Claim 3 yields (2.9).

Case 1.3.2. $r = m_0 = 0$. In this case since $Z(F, [a, b]) = Z(f - P, [a, b]) = N + 1$, by Rolle Theorem for every j, $j = 0, 1, ..., N$, there must exist the largest zero ξ_j of $F^{(j)}$ such that

$$x_1 = \xi_0 = \cdots = \xi_{m_1 - 1} > \xi_{m_1} > \cdots > \xi_N$$

and

$$F^{(j)}(\xi_j)=0, \qquad Z(F^{(j)}, (\xi_j, b])=0, \qquad j=0, 1, ..., N. \tag{2.10}$$

Recalling that $F^{(N+1)}(x)=f^{(N+1)}(x)>0$ in $[a, b]$ by (2.2), it follows from (2.10) by induction that

$$F^{(j)}(x)>0, \qquad x>\xi_j, \qquad j=N, N-1, ..., 1, 0.$$

In particular $F(x)>0$ for $x\in(x_1, b)$, which implies that

$$\operatorname{sgn}[\sigma(x)\, F(x)]=1=s_0$$

holds for $x\in(x_1, b)$ and hence yields (2.9) with $r=0$.

This completes the proof of Claim 4. With the help of (2.4) and (2.9) we get (1.7) and (1.8) under the assumption (2.2).

Part 2. The proof of (1.9) and (1.10). In this case put

$$P(x)=\sum_{k=r}^{n+1}\sum_{j=0}^{m_k-1} f^{(j)}(x_k)\, A_{jk}(x)$$

instead of (2.3) and distinguish two cases.

Case 2.1. $r=n+1$ and $m_{n+1}=0$. In this case (1.9) and (1.10) are trivial, because $P=0$.

Case 2.2. $r\leqslant n$ or $m_{n+1}>0$. In this case choose

$$F(x)=\begin{cases} f(x)-P(x), & x\leqslant x_r, \\ P(x), & x>x_r \end{cases} \tag{2.11}$$

instead of (2.4). By the same argument as above we again conclude that the function σF does not change sign in $[a, b]$. But in this case we claim

$$s_{r-1}\sigma(x)\, F(x)\geqslant 0, \qquad x\in[a, b]. \tag{2.12}$$

To prove (2.11) we again separate two cases.

Case 2.2.1. $r\geqslant 1$. In this case by the same argument as that of Case 1.3.1 we can obtain that $F(x)=P(x)>0$ for $x\in(x_r, x_{r-1})$ and hence by (2.1) $\operatorname{sgn}[\sigma(x)\, F(x)]=\sigma(x)=s_{r-1}$ holds for $x\in(x_r, x_{r-1})$, which yields (2.12).

Case 2.2.2. $r=0$. In this case (1.10) is trivial and it suffices to show (1.9). If $m_0=0$ then (1.9) is equivalent to (1.7). So it is enough to prove

(1.9) for the case when $m_0>0$. Since $Z(F,[a,b])=Z(f-P,[a,b])=N+1$, by Rolle Theorem for every j, $j=0, 1, ..., N$, there must exist the largest zero η_j of $F^{(j)}$ such that

$$x_0=\eta_0= \cdots =\eta_{m_0-1}>\eta_{m_0}> \cdots >\eta_N$$

and

$$F^{(j)}(\eta_j)=0, \qquad Z(F^{(j)},(\eta_j,b])=0, \qquad j=0, 1, ..., N. \tag{2.13}$$

From the fact $F^{(N+1)}(x)=f^{(N+1)}(x)>0$ in $[a,b]$ by (2.2), it follows from (2.13) by induction that

$$F^{(j)}(x)>0, \qquad x>\eta_j, \quad j=N, N-1, ..., m_0+1, m_0,$$

and further for small $\delta>0$

$$(-1)^{m_0-j}F^{(j)}(x)>0, \qquad x\in(b-\delta,b), \quad j=m_0-1, m_0-2, ..., 1, 0.$$

In particular $(-1)^{m_0}F(x)>0$ for $x\in(x_1,b)$, which implies that

$$\operatorname{sgn}[\sigma(x)\,F(x)]=(-1)^{m_0}(-1)^{m_0}=1=s_{-1}$$

holds for $x\in(x_1,b)$. Hence (2.12) with $r=0$ follows. This completes the proof of (2.12). With the help of (2.11) and (2.12) we directly get (1.9) and (1.10) under the assumption (2.2).

Next, if f satisfies (1.6) only, then we consider the function $f_\varepsilon(x)=f(x)+\varepsilon e^x$, $\varepsilon>0$, which already satisfies (2.2). Applying the above conclusion to the function f_ε and letting $\varepsilon\to0$ yields (1.7)–(1.10). ∎

3. CONSEQUENCES

Theorem 1 is very general and useful. We state some useful corollaries.

Corollary 1. *Let* $0\leqslant r<q\leqslant n+1$ *and let the* $(N+1)$*th continuously differentiable function* f *satisfy* (1.6). *If*

$$s_{r-1}=s_q \tag{3.1}$$

or equivalently

$$\sum_{k=r}^{q} m_k\in\mathbf{N}_2, \tag{3.2}$$

then

$$s_q\sigma(x)\left[f(x)-\sum_{k=r}^{q}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\right]\geqslant 0,\qquad x\in\langle x_q, x_r\rangle. \tag{3.3}$$

Proof. Clearly by (1.4) the equality (3.1) is equivalent to (3.2). The inequality (1.8) with $r=q$ reads

$$s_q\sigma(x)\sum_{k=q+1}^{n+1}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\geqslant 0,\qquad x\in\langle x_q, b\rangle,$$

which, together with (1.9) and (3.1), yields (3.3). ∎

COROLLARY 2. *Let* $0\leqslant r\leqslant n$ *and let an* $(N+1)$*th continuously differentiable function* f *satisfy* (1.6). *If*

$$m_r+m_{r+1}\in\mathbf{N}_2, \tag{3.4}$$

then

$$(-1)^{m_{r+1}}\left[f(x)-\sum_{k=r}^{r+1}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\right]\geqslant 0,\qquad x\in\langle x_{r+1}, x_r\rangle. \tag{3.5}$$

In particular, if

$$m_r, m_{r+1}\in\mathbf{N}_1, \tag{3.6}$$

then

$$\sum_{k=r}^{r+1}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\geqslant f(x),\qquad x\in\langle x_{r+1}, x_r\rangle. \tag{3.7}$$

Proof. By Corollary 1 with $q=r+1$ we obtain

$$s_{r+1}\sigma(x)\left[f(x)-\sum_{k=r}^{r+1}\sum_{j=0}^{m_k-1}f^{(j)}(x_k)\,A_{jk}(x)\right]\geqslant 0,\qquad x\in\langle x_{r+1}, x_r\rangle. \tag{3.8}$$

But by (2.1) for $x\in(x_{r+1}, x_r)$ we have $s_{r+1}\sigma(x)=s_{r+1}s_r=(-1)^{m_{r+1}}$ and hence (3.8) becomes (3.5). The relation (3.7) is an immediate consequence of (3.5) if (3.6) is true. ∎

Remark 1. For Lagrange interpolation the inequality (3.7) with $f=1$ becomes (1.3).

COROLLARY 3. *Let* $0 \leqslant r < q \leqslant n+1$ *and let an* $(N+1)$*th continuously differentiable function* f *satisfy* (1.6). *If*

$$m_k \in \mathbf{N}_2, \qquad k = r, r+1, ..., q, \tag{3.9}$$

then

$$\sum_{k=r}^{q} \sum_{j=0}^{m_k - 1} f^{(j)}(x_k)\, A_{jk}(x) \leqslant f(x), \qquad x \in \langle x_q, x_r \rangle. \tag{3.10}$$

Proof. Apply Corollary 1. ▮

COROLLARY 4. *Let* $0 \leqslant r \leqslant n+1$ *and let an* $(N+1)$*th continuously differentiable function* f *satisfy* (1.6). *If*

$$m_r \in \mathbf{N}_1, \tag{3.11}$$

then

$$s_{r-1}\sigma(x)\, sgn(x - x_r) \sum_{j=0}^{m_r - 1} f^{(j)}(x_r)\, A_{jr}(x) \geqslant 0, \qquad x \in \langle a, b \rangle. \tag{3.12}$$

Proof. The relation (3.11) means $s_r = -s_{r-1}$. Hence (1.7) and (1.9) give that the inequality

$$s_{r-1}\sigma(x)\, \operatorname{sgn}(x - x_r) \sum_{j=0}^{m_r - 1} f^{(j)}(x_r)\, A_{jr}(x) \geqslant 0 \tag{3.13}$$

holds for $x \in \langle a, x_r \rangle$; besides, (1.8) and (1.10) yield that the inequality (3.13) holds for $x \in \langle x_r, b \rangle$. ▮

COROLLARY 5. *Let* $d\mu$ *be a measure in* $\langle a, b \rangle$ *and let an* $(N+1)$*th continuously differentiable function* f *satisfy*

$$f^{(j)}(x) \geqslant 0, \qquad x \in \langle a, b \rangle, \quad j = 0, 1, ..., N+1. \tag{3.14}$$

Then

$$\sum_{k=0}^{n+1} \sum_{j=0}^{m_k - 1} C_{jk} f^{(j)}(x_k) \leqslant \int_a^b f(x)\, \sigma(x)\, d\mu(x), \tag{3.15}$$

where

$$C_{jk} = \int_a^b A_{jk}(x)\, \sigma(x)\, d\mu(x), \qquad j = 0, 1, ..., m_k - 1, \quad k = 0, 1, ..., n+1. \tag{3.16}$$

In particular, if the nodes x_k's are the solution of the extremal problem:

$$\int_a^b \left| \prod_{k=0}^{n+1} (x-x_k)^{m_k} \right| d\mu(x) = \min_{a=t_{n+1}<t_n<\cdots<t_1<t_0=b} \int_a^b \left| \prod_{k=0}^{n+1} (x-t_k)^{m_k} \right| d\mu(x), \tag{3.17}$$

then the inequality (3.15) *with*

$$C_{m_k-1,k}=0, \qquad k=1, 2, ..., n, \tag{3.18}$$

holds.

Proof. The inequality (3.15) follows from (1.9) with $r=0$. Further, if the relation (3.17) is true then by [1, Theorem 3] (3.18) must hold. ∎

Remark 2. The special case of the second part of Corollary 5 when $m_0=m_{n+1}=0$ and $m_1=\cdots=m_n=2$ can be found in [3, Lemma I.1.5, p. 92].

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